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AXISYMMETRIC VIBRATION OF LAMINATED ANNULAR PLATES COMPOSED OF TRANSVERSELY ISOTROPIC LAYERS

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1. INTRODUCTION

Thick and moderately thick annular plates composed of laminated composite materials are among the most important structural elements used in modern engineering. Such lightweight and highly reinforced components are also being increasingly used in civil, mechanical and transport engineering applications. Although a considerable amount of work has been reported in the area of axisymmetric vibration of annular plates, it has mainly been based on classic theories or two-dimensional shear deformable theories [1–4]. To the best of the author's knowledge, comparatively much less work, concerning three-dimensional analysis, is available in the literature.

Using three-dimensional elasticity theory, in this note an axisymmetric vibration analysis is presented of laminated annular plates composed of transversely isotropic layers. The analysis is based on a recursive method that has been successfully applied in connection with vibrations of laminated circular plates [5]. The three-dimensional governing equations for axisymmetric vibration of a transversely isotropic annular plate is first converted into a set of first order linear differential equation system for which analytical solutions can be obtained. This is achieved by introducing Bessel functions to decompose the governing equations. On the basis of the solutions obtained for each material layer, the solutions of the laminated plate are then obtained by imposing continuity conditions at all the interfaces and boundary conditions at the two lateral surfaces. Moreover, due to the use of the recursive formulation, regardless of the number of the material layers considered, the natural frequencies of a laminated annular plate are always obtained as the roots of a 2×2 eigen-determinant.

2. SOLUTION OF A SINGLE-LAYERED PLATE

Consider an annular plate with an arbitrary constant thickness h and denote by a and b its outer and inner radii, respectively. The radial, circumferential and transverse co-ordinates are denoted by r, θ and z, respectively, while the corresponding displacements in the r- and z-directions are represented by U and W. It is assumed that the plate is constructed of homogeneous transversely isotropic material. Accordingly, its axisymmetric dynamic equilibrium and elastic behaviour are described by the following equations under three-dimensional consideration:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = \rho \frac{\partial^2 w}{\partial t^2}, \tag{1}$$

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$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \begin{bmatrix} C_{rr} & C_{r\theta} & C_{rz} & 0 \\ C_{r\theta} & C_{rr} & C_{rz} & 0 \\ C_{rz} & C_{rz} & C_{zz} & 0 \\ 0 & 0 & 0 & G_{rz} \end{bmatrix} \begin{Bmatrix} \partial U/\partial r \\ U/r \\ \partial W/\partial z \\ \partial W/\partial r + \partial U/\partial z \end{Bmatrix}, \qquad (2)$$

where C_{ij} are the elastic constants of the material which has a density of ρ. Eliminating membrane stresses σ_r and σ_θ from equations (1) and (2) yields

$$\frac{\partial}{\partial z}\begin{Bmatrix} U \\ Z \\ R \\ W \end{Bmatrix} = \begin{bmatrix} 0 & 0 & C_5 & -\alpha \\ 0 & 0 & -(\alpha + 1/r) & \xi^2 \\ \xi^2 - C_2(\alpha^2 + \alpha/r - 1/r^2) & -C_1\alpha & 0 & 0 \\ -C_1(\alpha + 1/r) & C_4 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U \\ Z \\ R \\ W \end{Bmatrix}, \qquad (3)$$

where

$$\alpha = \partial/\partial r, \qquad \xi^2 = \rho\partial^2/\partial t^2, \qquad R = \tau_{rz}, \qquad Z = \sigma_z, \qquad C_1 = -C_{rz}/C_{zz},$$

$$C_2 = C_{rr} - C_{rz}/C_{zz}, \qquad C_3 = C_{r\theta} - C_{rz}/C_{zz}, \qquad C_4 = 1/C_{zz}, \qquad C_5 = 1/G_{rz}.$$

For axisymmetric vibrations of an annular plate, the displacements and stresses are defined as follows:

$$U = [\mathrm{d}f(r)/\mathrm{d}r]\bar{U}(z)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad W = f(r)\bar{W}(z)\,\mathrm{e}^{\mathrm{i}\omega t},$$

$$Z = f(r)\bar{Z}(z)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad R = [\mathrm{d}f(r)/\mathrm{d}r]\bar{R}(z)\,\mathrm{e}^{\mathrm{i}\omega t}. \qquad (4)$$

where $f(r)$ in equation (4) is a function which satisfies the Bessel equation

$$(\alpha^2 + \alpha/r + k^2)(\alpha^2 + \alpha/r - k^2)f(r) = 0, \qquad (5)$$

where k^2 is a constant that can be determined solely by considering boundary conditions imposed at the two edges of an annular plate.

Suppose that the value of k^2 for given boundary conditions has been determined. Using equation (5), the following matrix differential equation can be obtained by inserting equations (4) into (3):

$$\mathrm{d}\{\mathbf{F}\}/\mathrm{d}z = [\mathbf{G}]\{\mathbf{F}\}, \qquad \{\mathbf{F}\}^{\mathrm{T}} = [\bar{U}(z), \bar{Z}(z), \bar{R}(z), \bar{W}(z)]. \qquad (6)$$

The non-zero elements of $[\mathbf{G}]$ are given as follows:

$$G_{13} = C_5, \qquad G_{14} = -1, \qquad G_{23} = k^2, \qquad G_{24} = -\rho\omega^2,$$

$$G_{31} = C_2k^2 - \rho\omega^2, \qquad G_{32} = C_1, \qquad G_{41} = -C_1k^2, \qquad G_{42} = C_4. \qquad (7)$$

The general solution of equation (6) can be explicitly expressed as

$$\{\mathbf{F}(z)\} = [\mathbf{D}(z)]\{\mathbf{F}(-h/2)\}, \qquad -h/2 \leqslant z \leqslant h/2, \qquad (8)$$

where $\{\mathbf{F}(-h/2)\}$ denotes the value of the vector $\{\mathbf{F}\}$ at the bottom surface of the plate. For a given value of z (representing a z-surface that is parallel to the middle surface), the elements of the 4×4 matrix $[\mathbf{D}(z)] = \exp([\mathbf{G}]z)$ can be evaluated analytically in various ways. For a free vibration problem, imposing $\sigma_z(\pm h/2) = \tau_{rz}(\pm h/2) = 0$ to equation (8) yields the following equation system,

$$\begin{Bmatrix} \bar{U}(h/2) \\ \bar{W}(h/2) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \begin{Bmatrix} \bar{U}(-h/2) \\ \bar{W}(-h/2) \\ 0 \\ 0 \end{Bmatrix}, \tag{9}$$

and the bottom half of equation (9) gives

$$\begin{bmatrix} D_{31} & D_{32} \\ D_{41} & D_{42} \end{bmatrix} \begin{Bmatrix} \bar{U}(-h/2) \\ \bar{W}(-h/2) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{10}$$

where the D_{ij} are the corresponding elements of $[\mathbf{D}(h/2)]$. The exact natural frequencies of the homogeneous annular plate considered are obtained as the roots of the following eigenequation:

$$\det \begin{vmatrix} D_{31} & D_{32} \\ D_{41} & D_{42} \end{vmatrix} = 0. \tag{11}$$

3. SOLUTION OF A LAMINATED ANNULAR PLATE

For free vibration of a laminated annular plate, the solution of the problem is based on the division of the plate into N material layers, each of which may be made of different transversely isotropic materials and may have different thicknesses. For each sub-layer, a solution can be obtained in the form of equation (8). The solutions obtained for each of the sub-layers are then connected by imposing continuity conditions at the material interfaces and boundary conditions at the two lateral surfaces. Dealing in particular with the interface of the $(j-1)$th and jth of the aforementioned layers, having thicknesses h_{j-1} and h_j, respectively, the continuity conditions of displacements and transverse stresses lead to the following relation ($j = 2, 3, \ldots, N$),

$$\{\mathbf{F}^{(j)}(-h_j/2)\} = \{\mathbf{F}^{(j-1)}(-h_{j-1}/2)\}. \tag{12}$$

Hence, upon recursively using equations (8) and (12), the solution of the problem considered in such a N-layered composite plate can be obtained in the following form:

$$\begin{aligned} \{\mathbf{F}^{(N)}(h_N/2)\} &= [\mathbf{D}^{(N)}(h_N/2)]\{\mathbf{F}^{(N)}(-h_N/2)\} = [\mathbf{D}^{(N)}(h_N/2)]\{\mathbf{F}^{(N-1)}(h_{N-1}/2)\} \\ &= [\mathbf{D}^{(N)}(h_N/2)][\mathbf{D}^{(N-1)}(h_{N-1}/2)]\{\mathbf{F}^{(N-1)}(-h_{N-1}/2)\} \\ &= [\mathbf{D}^{(N)}(h_N/2)][\mathbf{D}^{(N-1)}(h_{N-1}/2)]\{\mathbf{F}^{(N-2)}(h_{N-2}/2)\} \\ &= \cdots \\ &= \left[\prod_{k=N}^{1} \mathbf{D}^{(k)}(h_k/2)\right]\{\mathbf{F}^{(1)}(-h_1/2)\}. \end{aligned} \tag{13}$$

This yields a linear algebraic eigenvalue problem of second order after applying stress free boundary conditions to the two lateral surfaces in the same manner as described below equation (9). It is obvious that, independently of the number of layers involved, the present formulation yields the solution of the problem always as the roots of a 2×2 eigen-determinant.

4. SOLUTION OF THE CONSTANT k^2

A solution of equation (5) is in the form

$$f(r) = A_1 J_0(kr) + A_2 N_0(kr) + A_3 I_0(kr) + A_4 K_0(kr), \tag{14}$$

where J_0 and N_0 are Bessel functions of the first and second kind of zero order, and I_0 and K_0 are the respective modified Bessel function of first and second kind of order zero. Introducing boundary conditions into equation (14) and seeking non-trivial solutions of the A_k ($k = 1, 2, 3, 4$), the values of k^2 can be obtained for various boundary conditions from the following equations [6].

(a) Clamped edge ($W = 0$, $U = 0$):

$$f(r) = 0, \qquad \mathrm{d}f(r)/\mathrm{d}r = 0. \tag{15}$$

(b) Hinged edge ($W = 0$, $\sigma_r = 0$):

$$f(r) = 0, \qquad \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right] f(r) = 0. \tag{16}$$

(c) Free edge

$$\sigma_r = 0, \qquad \int_{-h/2}^{h/2} \tau_{rz}\,\mathrm{d}z = 0,$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\nabla^2 f(r) = 0, \qquad \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right] f(r) = 0. \tag{17}$$

The k^2 obtained from the above equations [7] are then substituted into equations (6) and (7) to find the corresponding natural frequencies.

5. NUMERICAL EXAMPLES

For the purpose of comparison with the results obtained from alternative studies, the non-dimensional natural frequency parameter,

$$\Omega = \omega a(\rho/G)^{1/2}, \tag{18}$$

of a clamped isotropic annular plate ($v = 0{\cdot}3$) are first calculated. The comparisons are given in Table 1 for plates having differing thickness-to-radius ratios. All the results obtained by the present analysis are smaller than the corresponding ones by using the Method of Initial Function [6] and using Mindlin theory. Detailed study of these results shows that the discrepancies increase as the thickness-to-radius ratio increases. The natural frequencies for a thick plate ($h/a = 0{\cdot}2$), for which no comparisons can be made, are also given in Table 1.

The first three natural frequency parameters, Ω, for an isotropic annular plate against either the thickness-to-radius ratio, h/a, or the ratio between the inner and outer radii are presented in Table 2. The plate is clamped at the outer and free at the inner edges. As expected, the natural frequencies increase as h/a and b/a increases. Moreover, it is observed that the effect of the ratio b/a upon the fundamental frequencies is less significant than upon the higher mode ones.

TABLE 1

The first three frequency parameters Ω, for axisymmetric vibration of a clamped isotropic annular plate

h/a	Mode	Present	MIF	Mindlin
0·05	1	0·8303	0·8306	0·8367
	2	2·2440	2·2511	2·2624
	3	4·2578	4·2610	4·2971
0·10	1	1·5955	1·6153	1·6293
	2	4·0611	4·0977	4·1341
	3	7·2223	7·2519	7·3031
0·20	1	2·7951	—	—
	2	6·2967	—	—
	3	9·8250	—	—

Dealing with free vibration of laminated annular plates, in Table 3 are given the first three natural frequency parameters

$$\Omega^* = \omega a(\rho/G_1)^{1/2} \tag{19}$$

of sandwich annular plates made of three isotropic layers, where G_1 is the shear modulus of first layer; i.e., the bottom layer. The plates consist of two identical skin layers and the material properties of the skins and core layer are distinguished by the parameter

$$\beta = E_c/E_s = 0{\cdot}1, \tag{20}$$

which is the ratio between the elastic modulus of the core and skin layers. The thickness ratio of the plate is $h_c/h_s = 2$. For antisymmetric laminated annular plates, in Table 4 are presented results for annular plates having two, four and eight layers. All the layers have the same thicknesses and are also made of isotropic materials. The plates are constructed in such a way that

$$E_1 = E_3 = \cdots = E_{2k-1} \quad \text{and} \quad E_{2k}/E_{2k-1} = 10, \qquad k = 1, 2, 3, \ldots . \tag{21}$$

TABLE 2

The first three frequency parameters Ω, for axisymmetric vibration of an isotropic annular plate with clamped outer and free inner edges

		b/a		
h/a	Mode	0·1	0·2	0·3
0·05	1	0·2384	0·2537	0·3290
	2	0·9477	1·0305	1·5921
	3	2·0138	2·4520	4·0217
0·10	1	0·4871	0·4988	0·6483
	2	1·8087	1·9585	2·9552
	3	3·6764	4·4047	6·8686
0·20	1	0·9285	0·9496	1·2183
	2	3·1262	3·3550	4·8035
	3	5·7884	6·7422	9·7544

TABLE 3

The first three frequency parameters Ω, for axisymmetric vibration of symmetrically laminated annular plates with clamped edges*

		b/a		
h/a	Mode	0·1	0·2	0·4
0·05	1	0·5894	0·7376	1·2524
	2	1·4897	1·8308	2·9424
	3	2·6381	3·1727	4·8858
0·10	1	1·0219	1·2434	1·9556
	2	2·2615	2·6845	3·9848
	3	3·6239	4·2458	6·1737

It is apparent from Table 4 that the natural frequencies increase with increasing the number of layers. This observation, which is well-known from corresponding dynamic analyses based on two-dimensional plate theories, indicates that bending–extensional coupling due to lamination dies out with an increasing number of layers of an antisymmetric laminate.

6. CONCLUSIONS

A method suitable for three-dimensional axisymmetric vibration of homogeneous and laminated thick annular plates has been presented. On the basis of a recursive formulation, this method always makes use of a 2×2 frequency determinant, independently of the layer number of the laminates.

Numerical results have been given and discussed for plates having different lay-ups, thickness-to-radius ratios and ratios between the inner and outer radii. It is believed that the results presented here may be used as a benchmark for evaluating and assessing the accuracy of the predictions based on two-dimensional plate theories in use at present.

TABLE 4

The fundamental frequency parameters Ω, for axisymmetric vibration of antisymmetrically laminated annular plates with clamped edges*

		Lay-ups		
h/a	b/a	Two-layered	Four-layered	Eight-layered
0·05	0·1	1·0510	1·3916	1·4743
	0·2	1·3742	1·7749	1·8990
	0·4	2·4521	3·0874	3·2010
0·10	0·1	2·1198	2·5956	2·7377
	0·2	2·6617	3·2073	3·3877
	0·4	4·6083	5·2601	5·5571

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